Int. J. Solids Structures, 1967, Vol. 3, pp. 533 to 542. Pergamon Press Ltd. Printed in Great Britain

A LINEARIZED THEORY OF VISCO-PLASTICITY

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Abstract—A linearization of the constitutive equations for an incompressible visco-plastic solid is obtained from a complementary function based on the Tresca yield condition. The strain rates given by the linearization are independent of the intermediate principal stress. An analysis of the bending of a simply supported circular plate is given as an application.

INTRODUCTION

It has been suggested by Hill [1] that the unifying concept in the mechanics of solids is that of a convex function and in this investigation a convex function, the complementary function, is used to develop a linearized theory of visco-plasticity. Hill [1] has shown that extremum principles and uniqueness theorems can be obtained for a solid if two convex functions, the work function and the complementary function, exist for the solid.

Let σ_{ij} and e_{ij} be the components, with respect to rectangular axes $0x_i$, of the stress and strain rate tensors respectively. The work and complementary functions are given by†

$$E = \int \sigma_{ij}\,\mathrm{d}e_{ij},$$

and

$$E_c = \int e_{ij}\,\mathrm{d}\sigma_{ij},$$

respectively, and their sum,

$$E + E_c = \sigma_{ij} e_{ij},$$

is the rate of energy dissipation per unit volume. The stress and strain rate components are obtained from E and E_c as follows,

$$\sigma_{ij} = \frac{\partial E}{\partial e_{ij}}, \qquad e_{ij} = \frac{\partial E_c}{\partial \sigma_{ij}}, \tag{1a, b}$$

where the partial differentiations are with respect to the e_{ij} and σ_{ij} each taken as nine independent variables. If the medium is incompressible E_c is independent of σ_{kk} and σ_{ij} is replaced by the stress deviation s_{ij} in equations (1). According to equation (1b) the vector, in stress space, that represents the components of strain rate‡ is parallel to the normal to the surface of constant E_c at the corresponding stress point.

† The usual summation convention for a repeated letter suffix is used.

‡ The components of strain rate are multiplied by a scalar constant to obtain the dimensions of stress.

Simple examples of work and complementary functions are those for the Newtonian liquid,

$$E = \mu e_{ij} e_{ij} = \frac{\mu I}{2} \tag{2}$$

and

$$E_c = \frac{s_{ij} s_{ij}}{4\mu} = \frac{J}{2\mu} \tag{3}$$

where $I = 2e_{ij}e_{ij}$ and $J = (1/2)s_{ij}s_{ij}$ are invariants of the strain rate and stress deviation tensors and μ is a coefficient of viscosity. The constitutive equations for the Newtonian liquid may be obtained from (2) or (3) by using (1) are

$$e_{ij} = \frac{s_{ij}}{2\mu}.$$

Equation (3) is a special case of a complementary function

$$E_c = f(J), \tag{4}$$

where $f(J)$ denotes a function of the invariant J. The strain rates obtained from (4) by using (1b) are

$$e_{ij} = f'(J)s_{ij}. \tag{5}$$

Constitutive equations of the form (5) have been applied to creep problems [2].

In principal stress space, with the principal stresses σ_1, σ_2 and σ_3, taken as rectangular Cartesian co-ordinates, a surface of constant E_c given by (3) or (4) is a circular cylinder with its axis passing through the origin and with direction cosines $[1/(\sqrt{3}), 1/(\sqrt{3}), 1/(\sqrt{3})]$.

A Bingham solid [3] has the following constitutive equation for a state of simple shear stress $\sigma_{12} = \sigma_{21}$,

$$2\mu e_{12} = \left\langle 1 - \frac{K}{|\sigma_{12}|} \right\rangle \sigma_{12} \tag{6}$$

where μ is a coefficient of viscosity, K the yield stress in pure shear, and the angle brackets have the significance,

$$\langle F \rangle = \begin{matrix} 0 \text{ for } F < 0 \\ F \text{ for } F \geqslant 0 \end{matrix} .$$

Prager [4] has generalized equation (6) for arbitrary states of stress, and the solid described has as limiting cases the von Mises perfectly plastic solid and the Newtonian liquid. This solid has the following work and complementary functions,

$$E = \frac{\mu I}{2} + K(\sqrt{I}) \qquad \text{if } I \neq 0 \tag{7}$$

and

$$E_c = \frac{\langle (\sqrt{J}) - K \rangle^2}{2\mu} \tag{8}$$

where K is the yield stress in pure shear for the Mises yield condition. The constitutive equations obtained from (7) and (8) by using (1) are

$$s_{ij} = 2\left(\mu + \frac{K}{\sqrt{I}}\right)e_{ij} \qquad \text{if } I \neq 0 \tag{9}$$

and

$$e_{ij} = \frac{1}{2\mu}\left\langle 1 - \frac{K}{\sqrt{J}}\right\rangle s_{ij}. \tag{10}$$

The work function (7) is equal to half the rate of viscous energy dissipation plus the rate of plastic energy dissipation per unit volume and the complementary function (8) is equal to half the rate of viscous energy dissipation per unit volume. A surface of constant E_c given by (8) is also a circular cylinder in principal stress space.

LINEARIZED THEORY

The yield surface of a Tresca plastic solid is a regular hexagonal prism in principal stress space and may be regarded as a piecewise linear approximation for the cylindrical von Mises yield surface. Similarly, families of cylindrical surfaces of constant E_c can be approximated by families of hexagonal prisms obtained from a continuous complementary function with piecewise continuous derivatives. For example the complementary function (3) for a Newtonian liquid can be approximated by

$$E_c = (\sigma_{max} - \sigma_{min})^2/(6\mu) \tag{11}$$

where σ_{max} and σ_{min} are the maximum and minimum principal stresses. Figure 1 shows the intersections, in principal stress space, of a plane σ_3 = const. and surfaces of constant E_c, given by (3) and (11).

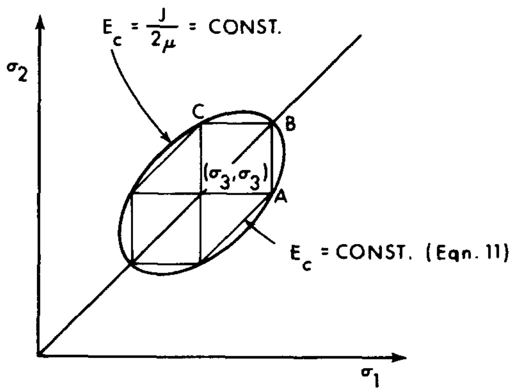

FIG. 1. Surfaces of constant E_c for viscous liquid.

According to (1b), if E_c is given by (11) and the stress point lies on a flat of the corresponding E_c surface, the strain-rate vector in principal stress space is normal to the flat. Consequently if the three principal stresses are distinct,

$$e_{max} = (\sigma_{max} - \sigma_{min})/(3\mu) = -e_{min}, \qquad e_{int} = 0 \tag{12a}$$

where e_{max}, e_{int} and e_{min} are the maximum, intermediate and minimum principal components of strain rate. If the stress point lies on a corner of the corresponding E_c surface given by (11), that is if two of the principal stresses are equal, the strain rate vector in principal stress space is not uniquely determined and may take any direction in the fan enclosed by the normals to the two intersecting flats at the corner. The complementary function (11) is equal to half the rate of energy dissipation per unit volume consequently the strain-rate vector corresponding to a stress point on a corner of the E_c surface must satisfy the requirement that the rate of energy dissipation per unit volume is the same for all admissible directions of the strain-rate vector. For example, referring to Fig. 1, the principal components of strain rate e_1, e_2 and e_3 corresponding to the stress point B are given by

$$\begin{aligned} e_1 &= t(\sigma_1 - \sigma_3)/(3\mu) \\ e_2 &= (1-t)(\sigma_2 - \sigma_3)/(3\mu) \\ e_3 &= -e_1 - e_2 \end{aligned} \tag{12b}$$

where t may take any value in the closed interval $0 \leqslant t \leqslant 1$. The approximation (11) for the complementary function (3) gives strain rates (12) that are not influenced by the intermediate principal stress and this is a simplification similar to that obtained by adopting the Tresca yield condition and its associated flow rule for a rigid plastic solid. Venkatraman and Hodge [5] have used a generalization of the stress–strain rate equations (12) to analyse creep in circular plates.

Prager [4] has noted that if a Newtonian liquid, a perfectly plastic von Mises solid and the visco-plastic Bingham solid with constitutive equations (9) and (10) are subjected to the same strain rate the stress in the Bingham solid is obtained by adding the stresses in the Newtonian liquid and the Mises solid. A visco-plastic solid such that for a given strain rate the stress is the sum of the stresses in a Tresca rigid perfectly plastic solid and a viscous liquid with its complementary function given by (11) is now considered. This visco-plastic solid has the following complementary function

$$E_c = \langle \sigma_{max} - \sigma_{min} - 2k \rangle^2/(6\mu) \tag{13}$$

where k is the yield stress in pure shear for the Tresca yield condition. The complementary function (13) may be regarded as an approximation for (8). Figure 2 shows the intersections, in principal stress space, of a plane $\sigma_3 = \text{const.}$ and surfaces of constant E_c given by (8) and (13). If the stress point lies on a flat of the corresponding E_c surface given by (13) the principal components of strain rate obtained from (1b) are

$$e_{max} = \langle \sigma_{max} - \sigma_{min} - 2k \rangle/(3\mu) = -e_{min}, \qquad e_{int} = 0. \tag{14a}$$

If the stress point lies on a corner the strain rates are obtained in the same manner as (12b). For example, referring to Fig. 2, the principal components of strain rate corresponding to point B are

$$\begin{aligned} e_1 &= t\langle \sigma_1 - \sigma_3 - 2k \rangle/(3\mu) \\ e_2 &= (1-t)\langle \sigma_2 - \sigma_3 - 2k \rangle/(3\mu) \\ e_3 &= -e_1 - e_2 \end{aligned} \tag{14b}$$

where t may take any value in the closed interval $0 \leqslant t \leqslant 1$. The complementary function given by (13), like that given by (8), is equal to half the rate of viscous energy dissipation

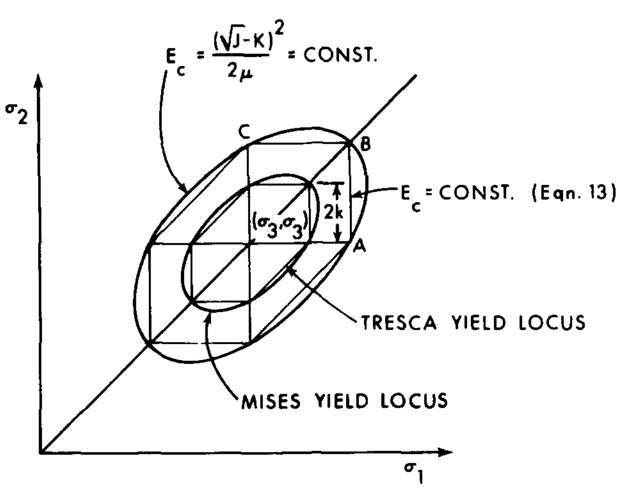

FIG. 2. Surfaces of constant E_c for visco-plastic solid.

per unit volume. The rate of energy dissipation per unit volume in terms of the stresses is

$$\dot{W}_\sigma = \frac{(\sigma_{\max} - \sigma_{\min})\langle \sigma_{\max} - \sigma_{\min} - 2k \rangle}{3\mu}$$

and in terms of the strain rates is

$$\dot{W}_e = 3\mu|e_m|^2 + 2k|e_m|$$

where $|e_m|$ is the magnitude of the numerically largest principal component of strain rate. The equation obtained from (14a) for a state of simple shear stress $\sigma_{12} = \sigma_{21}$ is

$$\frac{3\mu}{2} e_{12} = \left\langle 1 - \frac{k}{|\sigma_{12}|} \right\rangle \sigma_{12}\,. \tag{15}$$

Equation (15) is of the same form as equation (6), consequently the constitutive equations (14) are a valid generalization of the constitutive equation of a Bingham solid in simple shear.

Prager [6] has presented a different linearization of equations (10) which is based on a piecewise linear yield condition. When the Tresca yield condition is adopted the strain rates given by Prager's linearization are not always independent of the intermediate principal stress, but the constitutive equations of a Newtonian viscous liquid are obtained as a special case if the yield stress is zero.

BOUNDARY VALUE PROBLEM

Bending of a thin simply supported circular plate with a uniformly distributed transverse load is considered.

Let r, θ, z be cylindrical co-ordinates with the z-axis directed vertically downwards. The plate occupies the region $-h/2 \leqslant z \leqslant h/2$, $0 \leqslant r \leqslant a$, and is loaded with a uniformly distributed transverse load of intensity p acting in the z-direction. The equilibrium equation is

$$\frac{\mathrm{d}}{\mathrm{d}r}(rM_r)-M_\theta+\frac{pr^2}{2}=0 \tag{16}$$

where M_r and M_θ are the radial and circumferential principal components of bending moment and the principal curvature rates κ_r and κ_θ of the middle surface are given by†

$$\kappa_r=-\frac{\mathrm{d}^2w}{\mathrm{d}r^2},\qquad \kappa_\theta=-\frac{1}{r}\frac{\mathrm{d}w}{\mathrm{d}r} \tag{17a, b}$$

where w is the rate of deflection in the z-direction.

Since the plate is simply supported and symmetrically loaded, $M_r=M_\theta>0$ at $r=0$ and the boundary conditions are

$$M_r(a)=0,\qquad w(a)=0,\qquad \frac{\mathrm{d}w}{\mathrm{d}r}(0)=0. \tag{18a, b, c}$$

Also M_r, M_θ, w and $\mathrm{d}w/\mathrm{d}r$ are continuous throughout the plate.

Hopkins and Prager [7] have shown that the collapse intensity of loading if a Tresca rigid perfectly plastic plate is considered is

$$p_c=\frac{6M_0}{a^2}$$

where $M_0=kh^2/2$. Consequently flow of a visco-plastic plate, with its initial yielding governed by the Tresca yield condition, occurs if $p>p_c$. Figure 3 shows the yield locus for such a plate in terms of the principal bending moments M_r and M_θ. When $p=p_c$, for a Tresca rigid plastic plate, stress regime C–B (Fig. 3) applies for the entire plate with regime C for the edge $r=a$ and regime B for $r=0$. For visco-plastic flow the stress point lies outside the hexagon ABCDEF and it will be assumed that $M_\theta>M_r\geqslant 0$, for $\rho<r\leqslant a$ and $M_\theta=M_r>0$ for $0\leqslant r\leqslant\rho$, where ρ is a radius to be determined. Venkatraman and Hodge [5] have shown for the corresponding creep problem that this assumption is justified because of conditions (18a, c) and equation (16) and the reasoning used is also valid for the visco-plastic problem.

For the stress regimes of interest in the problem the complementary function expressed in terms of the principal bending moments is

$$\begin{aligned} E_c&=(1/2\lambda)\langle M_\theta-M_0\rangle^2 &&\text{if } M_\theta\geqslant M_r\\ E_c&=(1/2\lambda)\langle M_r-M_0\rangle^2 &&\text{if } M_r\geqslant M_\theta \end{aligned} \tag{19}$$

where λ is a coefficient of viscosity. The moment–curvature relations are obtained from (19) and the flow rule,

$$\kappa_r=\frac{\partial E_c}{\partial M_r},\qquad \kappa_\theta=\frac{\partial E_c}{\partial M_\theta},$$

† Bending moments and curvatures are assumed positive if they correspond to compression on the surface $z=-h/2$.

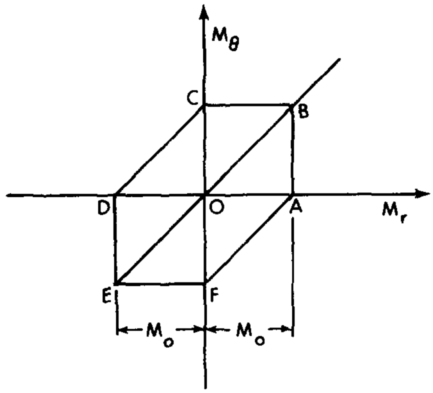

FIG. 3. Yield locus for circular plate.

which is the form that (1b) takes when expressed in terms of the principal moments, and are

$$\left.\begin{aligned}\lambda\kappa_r &= 0\\ \lambda\kappa_\theta &= \langle M_\theta - M_0\rangle\end{aligned}\right\} \rho \leqslant r \leqslant a \tag{20a, b}$$

$$\left.\begin{aligned}\lambda\kappa_r &= (1-t)\langle M_r - M_0\rangle\\ \lambda\kappa_\theta &= t\langle M_\theta - M_0\rangle\end{aligned}\right\} 0 \leqslant r \leqslant \rho \tag{21a, b}$$

where t takes values in the closed interval $0 \leqslant t \leqslant 1$. At $r = \rho$, $t = 1$ since κ_θ is continuous.

For the following analysis it is assumed that $p > p_c$.

Integration of equation (20a) and condition (18b) give

$$w = A(a-r) \qquad \rho \leqslant r \leqslant a \tag{22}$$

where A is a constant of integration.

Substitution of the curvature rate obtained from equations (17b) and (22) into equation (20b) results in,

$$M_\theta = \frac{A\lambda}{r} + M_0 \qquad \rho \leqslant r \leqslant a. \tag{23}$$

If equation (23) is substituted into the equilibrium equation (16) and the resulting equation is integrated using the boundary condition (18a) the following equation is obtained,

$$M_r = M_0\left(1-\frac{a}{r}\right) + \frac{A\lambda}{r}\ln\frac{r}{a} + \frac{pa^2}{6}\left(\frac{a}{r} - \frac{r^2}{a^2}\right) \qquad \rho \leqslant r \leqslant a. \tag{24}$$

Integration of equation (16) with $M_r = M_\theta$ gives

$$M_r = M_\theta = -\frac{pr^2}{4} + B \qquad 0 \leqslant r \leqslant \rho \tag{25}$$

where B is a constant of integration. Equations (17) and (21) may be expressed in the form

$$\lambda(\kappa_\theta + \kappa_r) = (M_r - M_0) = -\frac{\lambda}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}w}{\mathrm{d}r}\right)$$

which, combined with equation (25), gives

$$-\frac{\lambda}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right) = -\frac{pr^2}{4}+B-M_0$$

and integration and the condition (18c) give

$$w\lambda = \left(\frac{pr^4}{64}-\frac{Br^2}{4}+\frac{M_0 r^2}{4}\right)+C \qquad 0 \leqslant r \leqslant \rho \tag{26}$$

where C is a constant of integration. Since M_r and M_θ are continuous at $r = \rho$,

$$-\frac{1}{4}p\rho^2+B = \frac{\lambda A}{\rho}+M_0 \tag{27}$$

and

$$-\frac{1}{4}p\rho^2+B = M_0\left(1-\frac{a}{\rho}\right)+\frac{\lambda A}{\rho}\ln\frac{\rho}{a}+\frac{pa^2}{6}\left(\frac{a}{\rho}-\frac{\rho^2}{a^2}\right)$$

consequently

$$A = \frac{\dfrac{p\rho a^2}{6}\left(\dfrac{a}{\rho}-\dfrac{\rho^2}{a^2}\right)-M_0 a}{\lambda\left(1-\ln\dfrac{\rho}{a}\right)}\,. \tag{28}$$

Equations (22), (26), (27) and (28) and the requirement that dw/dr be continuous at $r = \rho$ give

$$\beta = (1-\alpha^3)-\frac{3}{4}\alpha^3(1-\ln\alpha) \tag{29}$$

where $\alpha = \rho/a$ and $\beta = 6M_0/pa^2 = p_c/p < 1$. For flow of a viscous plate with $M_0 = 0$, $\beta = 0$ and $\alpha = 0{\cdot}8056$. For a rigid plastic plate, loaded with the collapse load p_c, $\beta = 1$ and $\alpha = 0$. Consequently $0 < \alpha < 0{\cdot}8056$ for visco-plastic flow.

The following expressions for the moments are obtained from equations (23), (24), (25), (27) and (28),

$$\left.\begin{aligned}\frac{M_\theta}{M_0} &= \frac{3}{4}\frac{\alpha^3}{\beta}\frac{a}{r}+1\\ \frac{M_r}{M_0} &= \left(1-\frac{a}{r}\right)+\frac{3}{4}\frac{\alpha^3}{\beta}\frac{a}{r}\ln\frac{r}{a}+\frac{1}{\beta}\left(\frac{a}{r}-\frac{r^2}{a^2}\right)\end{aligned}\right\}\rho \leqslant r \leqslant a \tag{30}$$

$$\frac{M_r}{M_0} = \frac{M_\theta}{M_0} = \frac{9}{4}\frac{\alpha^2}{\beta}-\frac{3}{2\beta}\frac{r^2}{a^2}+1 \qquad 0 \leqslant r \leqslant \rho. \tag{31}$$

The deflection rates are obtained from equations (22) and (26) and the condition w is continuous at $r = \rho$ and are

$$\frac{w\lambda}{M_0 a^2} = \frac{3}{4}\frac{\alpha^3}{\beta}\left(1-\frac{r}{a}\right) \qquad \rho \leqslant r \leqslant a \tag{32}$$

$$\frac{w\lambda}{M_0 a^2} = \frac{3}{32\beta}\left(8\alpha^3 + \frac{r^4}{a^4} - 6\frac{\alpha^2 r^2}{a^2} - 3\alpha^4\right) \qquad 0 \leqslant r \leqslant \rho. \tag{33}$$

Equations (30), (31), (32) and (33) with equation (29) are a solution to the problem and are in a convenient form for numerical computation. It may be verified that the parameter t in equation (21) lies between 1 at $r = \rho$ and 0·5 at $r = 0$.

Appleby and Prager [8] solved the problem but used a different flow rule based on the linearization suggested by Prager [6].

CONCLUDING REMARKS

The theory presented in this paper can be extended to consider visco-plastic solids that have the following property. If a quasi-linear viscous liquid, a perfectly plastic von Mises solid, and the visco-plastic solid are subjected to the same strain rate, the stress in the visco-plastic solid is the sum of the stresses in the quasi-linear liquid and the von Mises solid. The complementary function and constitutive equations of the quasi-linear liquid are given by equations (4) and (5), respectively. For the visco-plastic solid the complementary function is

$$E_c = f(\phi) \tag{34}$$

where $f(\phi)$ denotes a function of ϕ and $\phi = \langle(\sqrt{J}) - K\rangle^2$ and the constitutive equations which may be found by applying (1b) are

$$e_{ij} = \{f'(\phi)\}\left\langle 1 - \frac{K}{\sqrt{J}}\right\rangle s_{ij}. \tag{35}$$

This visco-plastic solid described by equations (35) may be a realistic model for certain real solids if the function f is suitably chosen, but the complexity of the constitutive equations (35), even for the special case with $f(\phi) = \phi/2\mu$ already discussed, makes their application impractical except for trivial problems. However if the complementary function (34) is replaced by

$$E_c = f(\phi_1)$$

where

$$\phi_1 = (1/3)\langle\sigma_{\max} - \sigma_{\min} - 2k\rangle^2$$

some non-trivial problems, including the uniformly loaded simply supported plate problem, can be solved if f is a power function or an exponential function. Venkatraman and Hodge [5] used a similar theory to analyse creep of circular plates but did not use the concept of the complementary function. A creep theory can be obtained from a visco-plastic theory by letting the yield stress be zero.

Acknowledgement—This work was carried out as part of a project financed by a National Research Council of Canada grant in aid of research.

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(*Received* 19 *September* 1966; *revised* 10 *November* 1966)

Résumé—Une linéarisation des équations d'un solide incompressible visco-plastique est obtenue d'une fonction complémentaire basée sur la condition de rendement de Tresca. Le taux de déformation est donnée par la linéarisation et est indépendante du principe intermédiaire de tension. Une analyse de la flexion d'une plaque circulaire simplement supportée est donnée comme application.

Zusammenfassung—Eine lineare Darstellung der Zustandsgleichungen für nichtzusammendrückbare Festkörper wird erhalten, von der Komplementärfunktion die auf der Tresca'schen Fliessbedingung basiert. Die Spannungswerte der linearen Darstellung sind unabhängig von der Zwischenspannung. Eine Analyse der Biegung einer einfach gestützten Platte wird als Anwendungsbeispiel gegeben.

Абстракт—Получена линеаризация конститутивных уравнений для несжимаемого вязко-пластичного твёрдого тела из частного решения неоднородного линейного дифференциального уравнения, основанного на условии текучести Треска (Tresca). Скорости деформации, даваемые линеаризацией-независимы от промежуточного главного давления. Анализ изгибания свободно опёртой круговой пластины даётся, как приложение.